Exercise 30

Solve the telegraph equation in Exercise 29 with $V(x,0) = 0$ for

(a) the Kelvin ideal cable line ($L = 0 = G$) with the boundary data
$V(0,t) = V_0 = \text{const.}$, $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$.

(b) the noninductive leaky cable ($L = 0$) with the boundary conditions
$V(0,t) = H(t)$ and $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$.

Solution

Part (a)

When $L = 0$ and $G = 0$, the telegraph equation reduces to

$$-V_{xx} + RCV_t = 0.$$

Solving for V_t gives

$$V_t = \frac{1}{RC} V_{xx}.$$

Since we're given an initial condition and $t > 0$, this PDE can be solved with the Laplace transform. It is defined as

$$\mathcal{L}\{V(x,t)\} = \overline{V}(x,s) = \int_0^\infty e^{-st} V(x,t)\, dt,$$

which means the derivatives of V with respect to x and t transform as follows.

$$\mathcal{L}\left\{\frac{\partial^n V}{\partial x^n}\right\} = \frac{d^n \overline{V}}{dt^n}$$

$$\mathcal{L}\left\{\frac{\partial V}{\partial t}\right\} = s\overline{V}(x,s) - V(x,0)$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{V_t\} = \mathcal{L}\left\{\frac{1}{RC} V_{xx}\right\}$$

The Laplace transform is a linear operator, so the constant can be pulled out in front.

$$\mathcal{L}\{V_t\} = \frac{1}{RC}\mathcal{L}\{V_{xx}\}$$

Use the relations above to transform the partial derivatives.

$$s\overline{V}(x,s) - V(x,0) = \frac{1}{RC}\frac{d^2\overline{V}}{dx^2}$$

Since $V(x,0) = 0$, we just have (after multiplying both sides by RC)

$$\frac{d^2\overline{V}}{dx^2} = sRC\overline{V}.$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\overline{V}(x,s) = A(s)e^{\sqrt{sRC}x} + B(s)e^{-\sqrt{sRC}x}$$

In order for the condition, $V(x,t) \to 0$ as $x \to \infty$, to be satisfied, we require that $A(s) = 0$.

$$\overline{V}(x,s) = B(s)e^{-\sqrt{sRC}x}$$

To determine $B(s)$, we have to make use of the provided boundary condition, $V(0,t) = V_0$. Take the Laplace transform of both sides of it.

$$V(0,t) = V_0 \quad \to \quad \mathcal{L}\{V(0,t)\} = \mathcal{L}\{V_0\}$$

$$\overline{V}(0,s) = \frac{V_0}{s}$$

Plugging in $x = 0$ into the formula for $\overline{V}(x,s)$, we have

$$\overline{V}(0,s) = B(s) = \frac{V_0}{s}.$$

Thus,

$$\overline{V}(x,s) = \frac{V_0}{s}e^{-\sqrt{sRC}x}.$$

Now that we have $\overline{V}(x,s)$, all that's left to do is to take the inverse Laplace transform of it.

$$V(x,t) = \mathcal{L}^{-1}\{\overline{V}(x,s)\} = \mathcal{L}^{-1}\left\{\frac{V_0}{s}e^{-\sqrt{sRC}x}\right\}$$

Bring V_0 in front of the operator, and bring x under the square root.

$$V(x,t) = V_0\,\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-\sqrt{RCx^2s}}\right\}$$

This is a transform that can be looked up in a table.

$$V(x,t) = V_0 \operatorname{erfc}\left(\frac{\sqrt{RCx^2}}{2\sqrt{t}}\right),$$

where erfc is the complementary error function, a known special function, defined as

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-r^2}\,dr.$$

Therefore,

$$V(x,t) = V_0 \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right).$$

Part (b)

When $L = 0$, the telegraph equation reduces to

$$-V_{xx} + RCV_t + RGV = 0.$$

This PDE is first-order in the t variable, so we can use an integrating factor to simplify it. Divide both sides by RC.

$$-\frac{1}{RC}V_{xx} + V_t + \frac{G}{C}V = 0$$

Multiply both sides by the integrating factor,

$$I = e^{\int^t \frac{G}{C}\,ds} = e^{\frac{G}{C}t},$$

to get

$$-\frac{e^{\frac{G}{C}t}}{RC}V_{xx} + e^{\frac{G}{C}t}V_t + \frac{G}{C}e^{\frac{G}{C}t}V = 0.$$

The last two terms can be written as $\partial/\partial t(IV)$ as a result of the product rule. Also, since we're working with the partial derivatives of V, t is treated as a constant when taking the derivative with respect to x; hence, the exponential function can be brought inside the second derivative term.

$$-\frac{1}{RC}\frac{\partial^2}{\partial x^2}\left(e^{\frac{G}{C}t}V\right) + \frac{\partial}{\partial t}\left(e^{\frac{G}{C}t}V\right) = 0$$

Make the substitution,

$$W(x,t) = e^{\frac{G}{C}t}V(x,t),$$

so that the PDE simplifies to

$$-\frac{1}{RC}W_{xx} + W_t = 0,$$

which is the same one we solved in part (a). Taking the Laplace transform of both sides and solving the resulting ODE gives us

$$\overline{W}(x,s) = A(s)e^{\sqrt{RCs}x} + B(s)e^{-\sqrt{RCs}x}.$$

In order for the condition, $V(x,t) \to 0$ as $x \to \infty$, to be satisfied, we require that $A(s) = 0$.

$$\overline{W}(x,s) = B(s)e^{-\sqrt{sRC}x}$$

To determine $B(s)$, we have to make use of the provided boundary condition, $V(0,t) = H(t)$. Write $W(0,t)$ in terms of it and take the Laplace transform of both sides.

$$W(0,t) = e^{\frac{G}{C}t}V(0,t) = e^{\frac{G}{C}t}H(t) \quad \to \quad \mathcal{L}\{W(0,t)\} = \mathcal{L}\left\{e^{\frac{G}{C}t}H(t)\right\}$$

$$\overline{W}(0,s) = \frac{1}{s - \frac{G}{C}}$$

Plugging in $x = 0$ into the formula for $\overline{W}(x,s)$, we have

$$\overline{W}(0,s) = B(s) = \frac{1}{s - \frac{G}{C}}.$$

Thus,

$$\overline{W}(x,s) = \frac{1}{s - \frac{G}{C}} e^{-\sqrt{sRC}x}.$$

Now that we have $\overline{W}(x,s)$, we can change back to $W(x,t)$ by taking the inverse Laplace transform of it. Because we're taking the inverse Laplace transform of a product of functions, we can use the convolution theorem, which says

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The inverse Laplace transform of each individual function is

$$\mathcal{L}^{-1}\left\{\frac{1}{s - \frac{G}{C}}\right\} = e^{\frac{G}{C}t}$$

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{sRC}x}\right\} = \frac{x}{2}\sqrt{\frac{RC}{\pi t^3}} e^{-\frac{RCx^2}{4t}},$$

so we have

$$W(x,t) = \int_0^t e^{\frac{G}{C}(t-\tau)} \frac{x}{2}\sqrt{\frac{RC}{\pi\tau^3}} e^{-\frac{RCx^2}{4\tau}}\,d\tau.$$

Change back to $V(x,t)$ now.

$$e^{\frac{G}{C}t}V(x,t) = e^{\frac{G}{C}t}\int_0^t e^{-\frac{G}{C}\tau} \frac{x}{2}\sqrt{\frac{RC}{\pi\tau^3}} e^{-\frac{RCx^2}{4\tau}}\,d\tau$$

Cancel $e^{\frac{G}{C}t}$ from both sides to obtain

$$V(x,t) = \int_0^t e^{-\frac{G}{C}\tau} \frac{x}{2}\sqrt{\frac{RC}{\pi\tau^3}} e^{-\frac{RCx^2}{4\tau}}\,d\tau.$$

Pull the constants out in front and combine the exponential functions to obtain the final result.

$$V(x,t) = \frac{x}{2}\sqrt{\frac{RC}{\pi}} \int_0^t \frac{1}{\tau^{3/2}} e^{-\left(\frac{G}{C}\tau + \frac{RCx^2}{4\tau}\right)}\,d\tau$$

This answer is in disagreement with the answer at the back of the book. Interestingly, the answer there,

$$V(x,t) = \frac{1}{2}e^{-x\sqrt{b}}\operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{a}{t}} + \sqrt{\frac{bt}{a}}\right) + \frac{1}{2}e^{-x\sqrt{b}}\operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{a}{t}} - \sqrt{\frac{bt}{a}}\right),$$

does not satisfy the PDE. There must be a typo somewhere.